数理解析研究所講究録 第2095巻 2018年 48-56

On a maximizing problem of the Sobolev embedding related to the space of bounded variation

Michinori Ishiwata[1] and Hidemitsu Wadade[2]

[1] *Department of Systems Innovation, Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka 5608531, Japan*

[2] *Faculty of Mechanical Engineering, Institute of Science and Engineering, Kanazawa University, Kanazawa, Ishikawa 9201192, Japan*

1 Main theorem

We first recall the definition of the function space of bounded variation. Let $N\geq 2$. The total variation of $u\in L^{1}(\mathbb{R}^{N})$ is given by

$$V(u)(\mathbb{R}^{N}):=\sup\left\{\int_{\mathbb{R}^{N}}u\operatorname{div}\psi\,\middle|\,\psi\in C_{c}^{1}(\mathbb{R}^{N},\mathbb{R}^{N}),\ \|\psi\|_{L^{\infty}(\mathbb{R}^{N},\mathbb{R}^{N})}\leq 1\right\},$$

where $\|\psi\|_{L^{\infty}(\mathbb{R}^{N},\mathbb{R}^{N})}:=\max_{1\leq i\leq N}\|\psi_{i}\|_{L^{\infty}(\mathbb{R}^{N})}$ for $\psi=(\psi_{1},\cdots,\psi_{N})\in C_{c}^{1}(\mathbb{R}^{N},\mathbb{R}^{N})$. We say $u\in BV(\mathbb{R}^{N})$ if $u\in L^{1}(\mathbb{R}^{N})$ and $V(u)(\mathbb{R}^{N})<+\infty$.

Let $1<q\leq N'(:=\frac{N}{N-1})$ and $\alpha>0$. We consider the attainability of maximizing problems $D_{\alpha,q}$ and $\tilde{D}_{\alpha,q}$ defined by

$$D_{\alpha,q}:=\sup_{u\in BV(\mathbb{R}^{N}),\ \|u\|_{L^{1}(\mathbb{R}^{N})}+V(u)(\mathbb{R}^{N})=1}\left(\|u\|_{L^{1}(\mathbb{R}^{N})}+\alpha\|u\|_{L^{q}(\mathbb{R}^{N})}^{q}\right).$$

and

$$\tilde{D}_{\alpha,q}:=\sup_{u\in W^{1,1}(\mathbb{R}^{N}),\ \|u\|_{L^{1}(\mathbb{R}^{N})}+\|\nabla u\|_{L^{1}(\mathbb{R}^{N})}=1}\left(\|u\|_{L^{1}(\mathbb{R}^{N})}+\alpha\|u\|_{L^{q}(\mathbb{R}^{N})}^{q}\right).$$

Introduce the best-constant $GN_{q}>0$ of the Gagliardo-Nirenberg type inequality defined by

$$GN_{q}:=\sup_{u\in BV(\mathbb{R}^{N})\backslash\{0\}}GN_{q}(u):=\sup_{u\in BV(\mathbb{R}^{N})\backslash\{0\}}\frac{\|u\|_{L^{q}(\mathbb{R}^{N})}^{q}}{\|u\|_{L^{1}(\mathbb{R}^{N})}^{q-(q-1)N}V(u)(\mathbb{R}^{N})^{(q-1)N}}.$$

Also define $\alpha_{q}^{*}\geq 0$ by

$$\alpha_{q}^{*}:=\inf_{u\in BV(\mathbb{R}^{N}),\ \|u\|_{L^{1}(\mathbb{R}^{N})}+V(u)(\mathbb{R}^{N})=1}\frac{1-\|u\|_{L^{1}(\mathbb{R}^{N})}}{\|u\|_{L^{q}(\mathbb{R}^{N})}^{q}}.$$

Theorem 1.1 (Sub-critical case). *(i) When $1<q<\frac{N+1}{N}$, there holds $\alpha_{q}^{*}=0$, and $D_{\alpha,q}$ is attained for all $\alpha>0$. When $\frac{N+1}{N}\leq q<N'$, there holds $\alpha_{q}^{*}>0$, and $D_{\alpha,q}$ is attained for all $\alpha>\alpha_{q}^{*}$, while $D_{\alpha,q}$ is not attained for all $\alpha<\alpha_{q}^{*}$.*

(ii) When $q=\frac{N+1}{N}$, $D_{\alpha_{q}^{*},q}$ is not attained. When $\frac{N+1}{N}<q<N'$, $D_{\alpha_{q}^{*},q}$ is attained.

(iii) The values of α_q^* are computed as

$$\alpha_q^* = \begin{cases} 0 \quad \text{when } 1 < q < \frac{N+1}{N}, \\ \frac{1}{GN_q} \quad \text{when } q = \frac{N+1}{N}, \\ \frac{1}{GN_q} \frac{(q-1)^{q-1}}{(qN-(N+1))^{qN-(N+1)}(N-q(N-1))^{N-q(N-1)}}, \quad \text{when } \frac{N+1}{N} < q < N'. \end{cases}$$

(iv) There holds $GN_q = (\frac{1}{N^{N-1}\omega_{N-1}})^{q-1}$ for $1 < q \leq N'$.

Theorem 1.2 (Critical case). *There hold*

$$\alpha_{N'}^* = \frac{1}{GN_{N'}} = N\omega_{N-1}^{\frac{1}{N-1}} \quad \text{and} \quad D_{\alpha,N'} = \max\{1, \alpha\, GN_{N'}\},$$

and $D_{\alpha,N'}$ is not attained for all $\alpha > 0$.

Theorem 1.3. *Let $1 < q \leq N'$. Then there holds $D_{\alpha,q} = \tilde{D}_{\alpha,q}$, and $\tilde{D}_{\alpha,q}$ is not attained for all $\alpha > 0$.*

Theorem 1.4. *Assume one of the following conditions*

(i) $1 < q < \frac{N+1}{N}$ and $\alpha > 0$, (ii) $q = \frac{N+1}{N}$ and $\alpha > \alpha_q^*$, (iii) $\frac{N+1}{N} < q < N'$ and $\alpha \geq \alpha_q^*$.

Then there exists $R > 0$ depending on N, q and α such that the function

$$\frac{N}{\omega_{N-1}R^{N-1}(N+R)}\chi_{B_R(x_0)} \tag{1.1}$$

is a maximizer of $D_{\alpha,q}$ for all $x_0 \in \mathbb{R}^N$. Moreover, the function (1.1) is a unique maximizer of $D_{\alpha,q}$ except for the translation.

2 Preliminaries

Let $N \geq 2$ and $1 < q \leq N'$. Introduce the best-constants GN_q and $\tilde{GN}_q$ of the Gagliardo-Nirenberg type inequalities based on $BV(\mathbb{R}^N)$ and $W^{1,1}(\mathbb{R}^N)$ respectively by

$$GN_q := \sup_{u \in BV(\mathbb{R}^N)\setminus\{0\}} GN_q(u) \quad \text{and} \quad \tilde{GN}_q := \sup_{u \in W^{1,1}(\mathbb{R}^N)\setminus\{0\}} \tilde{GN}_q(u),$$

where

$$GN_q(u) := \frac{\|u\|_q^q}{\|u\|_1^{q-(q-1)N} V(u)^{(q-1)N}} \quad \text{for } u \in BV(\mathbb{R}^N) \setminus \{0\}$$

and

$$\tilde{GN}_q(u) := \frac{\|u\|_q^q}{\|u\|_1^{q-(q-1)N} \|\nabla u\|_1^{(q-1)N}} \quad \text{for } u \in W^{1,1}(\mathbb{R}^N) \setminus \{0\}.$$

Our goal in this section is to prove the following proposition.

Proposition 2.1. *Let $1 < q \leq N'$.*

(i) There holds $GN_q = \tilde{GN}_q = \left(\frac{1}{N^{N-1}\omega_{N-1}}\right)^{q-1}$.

(ii) GN_q is attained by functions of the form $u = \lambda\chi_B \in BV(\mathbb{R}^N)$ for $\lambda \in \mathbb{R} \setminus \{0\}$ and a ball $B \subset \mathbb{R}^N$. Moreover, the maximizer of GN_q necessarily has this form.

(iii) $\tilde{GN}_q$ is not attained in $W^{1,1}(\mathbb{R}^N) \setminus \{0\}$.

Proof. First, recall the facts that it holds $GN_{N'} = \frac{1}{N\omega_{N-1}^{\frac{1}{N-1}}}$ and $GN_{N'}$ is attained only by functions of the form $u = \lambda\chi_B \in BV(\mathbb{R}^N)$ for $\lambda \in \mathbb{R}\setminus\{0\}$ and a ball $B \subset \mathbb{R}^N$.

(i) By Hölder's inequality and Sobolev's inequality, we have for $u \in BV(\mathbb{R}^N)$

$$
\begin{aligned}
&\|u\|_q^q \le \|u\|_1^{q-(q-1)N}\|u\|_{N'}^{(q-1)N} \\
&\le \|u\|_1^{q-(q-1)N}\left(\frac{1}{N^{\frac{N-1}{N}}\omega_{N-1}^{\frac{1}{N}}}V(u)\right)^{(q-1)N} = \left(\frac{1}{N^{N-1}\omega_{N-1}}\right)^{q-1}\|u\|_1^{q-(q-1)N}V(u)^{(q-1)N},
\end{aligned}
$$

which implies $GN_q \le \left(\frac{1}{N^{N-1}\omega_{N-1}}\right)^{q-1}$. Let $u_0 = \chi_{B_1(0)} \in BV(\mathbb{R}^N)$. Then we can compute $\|u_0\|_1 = \|u_0\|_q^q = \frac{\omega_{N-1}}{N}$ and $V(u_0) = \omega_{N-1}$, and then we observe $GN_q(u_0) = \left(\frac{1}{N^{N-1}\omega_{N-1}}\right)^{q-1}$. Hence, u_0 is a maximizer of GN_q and it follows $GN_q = \left(\frac{1}{N^{N-1}\omega_{N-1}}\right)^{q-1}$.

Next, we prove $GN_q = \tilde{GN}_q$. It is enough to show $GN_q \le \tilde{GN}_q$ since the converse inequality is obtained by the facts $W^{1,1}(\mathbb{R}^N) \subset BV(\mathbb{R}^N)$ and $\|\nabla u\|_1 = V(u)$ for $u \in W^{1,1}(\mathbb{R}^N)$. Let $u_0 \in BV(\mathbb{R}^N)\setminus\{0\}$ be a maximizer of GN_q, where note that the existence of u_0 is already seen as above. By an approximation argument, there exists a sequence $\{u_n\}_{n=1}^\infty \subset BV(\mathbb{R}^N)\cap C^\infty(\mathbb{R}^N)$ such that $u_n \to u_0$ in $L^1(\mathbb{R}^N)$ and $V(u_n) \to V(u_0)$, and up to a subsequence, $u_n \to u_0$ a.e. on $\mathbb{R}^N$. We observe that $u_n \in W^{1,1}(\mathbb{R}^N)$ with $V(u_n) = \|\nabla u_n\|_1$. Indeed, by using the fact that there holds $V(v)(\Omega) = \int_\Omega |\nabla v|$ for any $v \in BV(\Omega)\cap C^\infty(\Omega)$ with a bounded domain having its sufficiently smooth boundary, we see

$$
V(u_n) = \sup_{R>0} V(u_n)(B_R) = \sup_{R>0}\int_{B_R}|\nabla u_n| = \lim_{R\to\infty}\int_{B_R}|\nabla u_n| = \|\nabla u_n\|_1 < +\infty,
$$

where the last equality is shown by Lebesgue's monotone convergence theorem. Then it holds $u_n \ne 0$ in $W^{1,1}(\mathbb{R}^N)$ for large $n \in \mathbb{N}$ since $\|\nabla u_n\|_1 = V(u_n) \to V(u_0) > 0$ as $n \to \infty$. Now we see by the convergences of u_n together with Fatou's lemma,

$$
GN_q = GN_q(u_0) \le \liminf_{n\to\infty} GN_q(u_n) \le \limsup_{n\to\infty} GN_q(u_n) = \limsup_{n\to\infty}\tilde{GN}_q(u_n) \le \tilde{GN}_q.
$$

Thus the assertion (i) has been proved.

(ii) Let $u_0 = \lambda\chi_B \in BV(\mathbb{R}^N)$ for $\lambda \in \mathbb{R}\setminus\{0\}$ and a ball $B = B_R(x_0)$ with a radius $R > 0$ centered at $x_0 \in \mathbb{R}^N$. Then we can compute

$$
\|u_0\|_1 = |\lambda|R^N\frac{\omega_{N-1}}{N}, \quad \|u_0\|_q^q = |\lambda|^q R^N\frac{\omega_{N-1}}{N} \quad \text{and} \quad V(u_0) = |\lambda|R^{N-1}\omega_{N-1},
$$

and thus these relations together with the assertion (i) show $GN_q(u_0) = \left(\frac{1}{N^{N-1}\omega_{N-1}}\right)^{q-1} = GN_q$. Hence, u_0 is a maximizer of GN_q.

Next, assume that GN_q is attained by $u_0 \in BV(\mathbb{R}^N)\setminus\{0\}$. Then by Hölder's inequality, Sobolev inequality and the assertion (i), we have

$$
\begin{aligned}
&\left(\frac{1}{N^{N-1}\omega_{N-1}}\right)^{q-1} = GN_q = GN_q(u_0) \\
&\le GN_{N'}(u_0)^{(q-1)(N-1)} \le GN_{N'}^{(q-1)(N-1)} = \left(\frac{1}{N^{N-1}\omega_{N-1}}\right)^{q-1},
\end{aligned}
$$

which shows that u_0 is a maximizer of $GN_{N'}$. Hence, $u_0 = \lambda\chi_B$ for some $\lambda \in \mathbb{R}\setminus\{0\}$ and a ball $B \subset \mathbb{R}^N$. The assertion (ii) has been proved.

(iii) By contradiction, assume that $\tilde{GN}_q$ is attained by $u_0 \in W^{1,1}(\mathbb{R}^N)\setminus\{0\}$. Then the assertion (i) and the facts $W^{1,1}(\mathbb{R}^N) \subset BV(\mathbb{R}^N)$ and $\|\nabla u\|_1 = V(u)$ for $u \in W^{1,1}(\mathbb{R}^N)$

imply that $u_0 \in BV(\mathbb{R}^N) \setminus \{0\}$ is a maximizer of GN_q. Then the assertion (ii) shows that $u_0 = \lambda \chi_B$ for $\lambda \in \mathbb{R} \setminus \{0\}$ and a ball $B \subset \mathbb{R}^N$, which is a contradiction to $u_0 \in W^{1,1}(\mathbb{R}^N)$. The assertion (iii) has been proved. $\square$

Proposition 2.2. *Let $1 < q \le N'$ and $\alpha > 0$. Then there hold*

$$D_{\alpha,q} = \sup_{t>0} f_\alpha(t) \quad \text{and} \quad \alpha_q^* = \frac{1}{GN_q} \inf_{t>0} g(t),$$

where

$$f_\alpha(t) := \frac{(1+t)^{q-1} + \alpha\, GN_q t^{(q-1)N}}{(1+t)^q} \quad \text{and} \quad g(t) := \frac{t(1+t)^{q-1}}{t^{(q-1)N}}$$

for $t > 0$. Furthermore, the values of α_q^* are computed as

$$\alpha_q^* = \begin{cases} 0 \quad \text{when } 1 < q < \frac{N+1}{N}, \\ \frac{1}{GN_q} \quad \text{when } q = \frac{N+1}{N}, \\ \frac{1}{GN_q} \frac{(q-1)^{q-1}}{(qN-(N+1))^{qN-(N+1)}(N-q(N-1))^{N-q(N-1)}} \quad \text{when } \frac{N+1}{N} < q < N', \\ \frac{1}{GN_q} \quad \text{when } q = N'. \end{cases}$$

Proof. For $u \in BV(\mathbb{R}^N)$ with $\|u\|_1 + V(u) = 1$, we see

$$\begin{aligned} \|u\|_1 + \alpha\|u\|_q^q &\le \|u\|_1 + \alpha\, GN_q \|u\|_1^{q-(q-1)N} V(u)^{(q-1)N} \\ &= \frac{\|u\|_1 (\|u\|_1 + V(u))^{q-1} + \alpha\, GN_q \|u\|_1^{q-(q-1)N} V(u)^{(q-1)N}}{(\|u\|_1 + V(u))^q} \\ &= \frac{\left(1 + \frac{V(u)}{\|u\|_1}\right)^{q-1} + \alpha\, GN_q \left(\frac{V(u)}{\|u\|_1}\right)^{(q-1)N}}{\left(1 + \frac{V(u)}{\|u\|_1}\right)^q} \\ &= f_\alpha\left(\frac{V(u)}{\|u\|_1}\right) \le \sup_{t>0} f_\alpha(t), \end{aligned}$$

which implies $D_{\alpha,q} \le \sup_{t>0} f_\alpha(t)$. On the other hand, let $v \in BV(\mathbb{R}^N)\setminus\{0\}$ be a maximizer of GN_q, where the existence of v is guaranteed by Proposition 2.1 (ii). For any $\lambda > 0$, let $v_\lambda(x) := \lambda v(\lambda^{\frac{1}{N}} x)$ and

$$w_\lambda(x) := \frac{v_\lambda(x)}{\|v_\lambda\|_1 + V(v_\lambda)} = \frac{\lambda v(\lambda^{\frac{1}{N}} x)}{\|v\|_1 + \lambda^{\frac{1}{N}} V(v)}.$$

Then for any $\lambda > 0$,

$$\begin{aligned} D_{\alpha,q} &\ge \|w_\lambda\|_1 + \alpha\|w_\lambda\|_q^q \\ &= \frac{\|v\|_1}{\|v\|_1 + \lambda^{\frac{1}{N}} V(v)} + \alpha \frac{\lambda^{q-1}\|v\|_q^q}{\left(\|v\|_1 + \lambda^{\frac{1}{N}} V(v)\right)^q} \\ &= \frac{\left(1 + \lambda^{\frac{1}{N}} \frac{V(v)}{\|v\|_1}\right)^{q-1} + \alpha\, GN_q \left(\lambda^{\frac{1}{N}} \frac{V(v)}{\|v\|_1}\right)^{(q-1)N}}{\left(1 + \lambda^{\frac{1}{N}} \frac{V(v)}{\|v\|_1}\right)^q} \\ &= f_\alpha\left(\lambda^{\frac{1}{N}} \frac{V(v)}{\|v\|_1}\right), \end{aligned}$$

which implies

$$D_{\alpha,q} \ge \sup_{\lambda>0} f_\alpha\left(\lambda^{\frac{1}{N}} \frac{V(v)}{\|v\|_1}\right) = \sup_{t>0} f_\alpha(t).$$

Thus there holds $D_{\alpha,q} = \sup_{t>0} f_\alpha(t)$.

Next, for $u \in BV(\mathbb{R}^N)$ with $\|u\|_1 + V(u) = 1$, we see

$$\begin{aligned}
\frac{1-\|u\|_1}{\|u\|_q^q} &\geq \frac{1-\|u\|_1}{GN_q\|u\|_1^{q-(q-1)N}V(u)^{(q-1)N}} \\
&= \frac{1}{GN_q}\frac{\left(\frac{V(u)}{\|u\|_1}\right)\left(1+\frac{V(u)}{\|u\|_1}\right)^{q-1}}{\left(\frac{V(u)}{\|u\|_1}\right)^{(q-1)N}} = \frac{1}{GN_q}g\left(\frac{V(u)}{\|u\|_1}\right) \geq \frac{1}{GN_q}\inf_{t>0} g(t),
\end{aligned}$$

which implies $\alpha_q^* \geq \frac{1}{GN_q}\inf_{t>0} g(t)$. On the other hand, let $v \in BV(\mathbb{R}^N)\backslash\{0\}$ be a maximizer of GN_q and define w_λ for $\lambda > 0$ as above. Then we see for $\lambda > 0$,

$$\begin{aligned}
\alpha_q^* &\leq \frac{1-\|w_\lambda\|_1}{\|w_\lambda\|_q^q} = \frac{\lambda^{\frac{1}{N}}V(v)\left(\|v\|_1+\lambda^{\frac{1}{N}}V(v)\right)^{q-1}}{\lambda^{q-1}\|v\|_q^q} \\
&= \frac{\lambda^{\frac{1}{N}}V(v)\left(\|v\|_1+\lambda^{\frac{1}{N}}V(v)\right)^{q-1}}{\lambda^{q-1}GN_q\|v\|_1^{q-(q-1)N}V(v)^{(q-1)N}} \\
&= \frac{\lambda^{\frac{1}{N}}\frac{V(v)}{\|v\|_1}\left(1+\lambda^{\frac{1}{N}}\frac{V(v)}{\|v\|_1}\right)^{q-1}}{GN_q\left(\lambda^{\frac{1}{N}}\frac{V(v)}{\|v\|_1}\right)^{(q-1)N}} = \frac{1}{GN_q}g\left(\lambda^{\frac{1}{N}}\frac{V(v)}{\|v\|_1}\right),
\end{aligned}$$

which implies

$$\alpha_q^* \leq \frac{1}{GN_q}\inf_{\lambda>0} g\left(\lambda^{\frac{1}{N}}\frac{V(v)}{\|v\|_1}\right) = \frac{1}{GN_q}\inf_{t>0} g(t).$$

Thus there holds $\alpha_q^* = \frac{1}{GN_q}\inf_{t>0} g(t)$.

Next, we compute the values of α_q^*. Since we have proved $\alpha_q^* = \frac{1}{GN_q}\inf_{t>0} g(t)$, it is enough to manipulate $\inf_{t>0} g(t)$. First, let $1 < q < \frac{N+1}{N}$. In this case, since $(N+1)-qN > 0$, we see $\inf_{t>0} g(t) = \inf_{t>0} t^{(N+1)-qN}(1+t)^{q-1} = 0$. Next, let $q = \frac{N+1}{N}$. In this case, since $(N+1)-qN = 0$, we see $\inf_{t>0} g(t) = \inf_{t>0}(1+t)^{q-1} = 1$. Next, let $\frac{N+1}{N} < q < N'$. In this case, we have $g(t) = \frac{(1+t)^{q-1}}{t^{qN-(N+1)}}$ with $qN-(N+1) > 0$ and

$$g'(t) = \frac{(1+t)^{q-2}}{t^{(q-1)N}}\left((N-q(N-1))\,t - (qN-(N+1))\right).$$

Then letting $t_0 := \frac{qN-(N+1)}{N-q(N-1)} > 0$, we obtain

$$\inf_{t>0} g(t) = g(t_0) = \frac{(q-1)^{q-1}}{(qN-(N+1))^{qN-(N+1)}\,(N-q(N-1))^{N-q(N-1)}}.$$

Finally, let $q = N'$. In this case, we have $g(t) = \frac{t(1+t)^{N'-1}}{t^{N'}}$ and $g'(t) = -\frac{N'-1}{t^{N'}(1+t)^{2-N'}} < 0$. Hence, we obtain $\inf_{t>0} g(t) = \lim_{t\to\infty} g(t) = 1$. The proof of Proposition 2.2 is complete. $\square$

3 Proof of main Theorems

Let $N \geq 2$. We start with the following lemma.

Lemma 3.1. *Let $1 < q < N'$.*

(i) Let $\alpha > \alpha_q^*$. Then $D_{\alpha,q}$ is attained.

(ii) Assume $\alpha_q^* > 0$ and let $0 < \alpha < \alpha_q^*$. Then $D_{\alpha,q}$ is not attained.

Proof. By Proposition 2.2, we see that $D_{\alpha,q}$ is attained if and only if $\sup_{t>0} f_\alpha(t)$ is attained.

(i) Let $\alpha > \alpha_q^*$. Note that the condition $q < N'$ shows $\lim_{t\to\infty} f_\alpha(t) = 0$. By the assumption $\alpha > \alpha_q^*$ and Proposition 2.2, there exists $t_0 > 0$ such that $\alpha > \frac{1}{GN_q} g(t_0)$, which implies $f_\alpha(t_0) > 1 = \lim_{t\downarrow 0} f_\alpha(t)$. Hence, $\sup_{t>0} f_\alpha(t)$ is attained.

(ii) Assume $\alpha_q^* > 0$ and let $0 < \alpha < \alpha_q^*$. By contradiction, assume that there exists $t_0 > 0$ such that $\sup_{t>0} f_\alpha(t) = f_\alpha(t_0)$. First, note $\sup_{t>0} f_\alpha(t) \geq \lim_{t\downarrow 0} f_\alpha(t) = 1$. By the assumption $\alpha < \alpha_q^*$ and Proposition 2.2, we obtain $\alpha < \alpha_q^* \leq \frac{1}{GN_q} g(t_0)$, which implies $f_\alpha(t_0) < 1$. Then we see $1 \leq \sup_{t>0} f_\alpha(t) = f_\alpha(t_0) < 1$, which is a contradiction. Thus $\sup_{t>0} f_\alpha(t)$ is not attained. $\square$

Proof of Theorem 1.1. By Proposition 2.1 (i), Proposition 2.2 and Lemma 3.1, it remains to prove that $D_{\alpha_q^*,q}$ is not attained when $q = \frac{N+1}{N}$, and $D_{\alpha_q^*,q}$ is attained when $\frac{N+1}{N} < q < N'$. First, let $q = \frac{N+1}{N}$. In this case, since $\alpha_q^* GN_q = 1$, we obtain $f_{\alpha_q^*}(t) = \frac{(1+t)^{\frac{1}{N}}+t}{(1+t)^{\frac{N+1}{N}}}$, and then $f'_{\alpha_q^*}(t) = \frac{N-t-N(1+t)^{\frac{1}{N}}}{N(1+t)^{\frac{2N+1}{N}}} < 0$ for all $t > 0$. Hence, $\sup_{t>0} f_{\alpha_q^*}(t)$ is not attained. Next, let $\frac{N+1}{N} < q < N'$. In this case, since $\lim_{t\downarrow 0} g(t) = \lim_{t\to\infty} g(t) = \infty$, there exists $t_0 > 0$ such that $\alpha_q^* = \frac{1}{GN_q} \inf_{t>0} g(t) = \frac{1}{GN_q} g(t_0)$, which gives $f_{\alpha_q^*}(t_0) = 1$. On the other hand, by noticing $\lim_{t\to\infty} f_{\alpha_q^*}(t) = 0$ by the condition $q < N'$ together with $\lim_{t\downarrow 0} f_{\alpha_q^*}(t) = 1$, we see that $\sup_{t>0} f_{\alpha_q^*}(t)$ is attained. Thus Theorem 1.1 has been proved. $\square$

Proof of Theorem 1.2. By Proposition 2.2, we already proved $\alpha_{N'}^* = \frac{1}{GN_{N'}}$. Hence, we show $D_{\alpha,N'} = \max\{1, \alpha GN_{N'}\}$, and $D_{\alpha,N'}$ is not attained for all $\alpha > 0$. In this case, we have

$$f_\alpha(t) = \frac{(1+t)^{N'-1} + \alpha GN_{N'} t^{N'}}{(1+t)^{N'}} \quad \text{and} \quad f'_\alpha(t) = \frac{t^{N'-1}}{(1+t)^{N'+1}} \left(\alpha N' GN_{N'} - \left(\frac{1+t}{t} \right)^{N'-1} \right).$$

We distinguish between two cases. When $\alpha \leq \frac{1}{N'GN_{N'}}$, we obtain $f'_\alpha(t) < 0$ for all $t > 0$, and hence, $\sup_{t>0} f_\alpha(t)$ is not attained. Also, in this case, we see $D_{\alpha,N'} = \sup_{t>0} f_\alpha(t) = \lim_{t\downarrow 0} f_\alpha(t) = 1 = \max\{1, \alpha GN_{N'}\}$. When $\alpha > \frac{1}{N'GN_{N'}}$, by putting $t_0 := \frac{1}{(\alpha N' GN_{N'})^{\frac{1}{N'-1}} - 1} > 0$, we see that f_α is strictly decreasing in $(0, t_0)$ and strictly increasing in (t_0, ∞), and therefore, $\sup_{t>0} f_\alpha(t)$ is not attained. Also, in this case, we see $D_{\alpha,N'} = \sup_{t>0} f_\alpha(t) = \max\{\lim_{t\downarrow 0} f_\alpha(t), \lim_{t\to\infty} f_\alpha(t)\} = \max\{1, \alpha GN_{N'}\}$. The proof of Theorem 1.2 is complete. $\square$

Proof of Theorem 1.3. Let $1 < q \leq N'$ and $\alpha > 0$. First, we prove $D_{\alpha,q} = \tilde{D}_{\alpha,q}$. It is enough to show $D_{\alpha,q} \leq \tilde{D}_{\alpha,q}$ since the converse inequality is obtained by the facts $W^{1,1}(\mathbb{R}^N) \subset BV(\mathbb{R}^N)$ and $\|\nabla u\|_1 = V(u)$ for $u \in W^{1,1}(\mathbb{R}^N)$. By the definition of $D_{\alpha,q}$, for any $\varepsilon > 0$, there exists $u_0 \in BV(\mathbb{R}^N)$ such that $\|u_0\|_1 + V(u_0) = 1$ and $\|u_0\|_1 + \alpha \|u_0\|_q^q > D_{\alpha,q} - \varepsilon$. As in the proof of Proposition 2.1 (i), we can pick up a sequence $\{u_n\}_{n=1}^\infty \subset W^{1,1}(\mathbb{R}^N)$ satisfying $u_n \to u_0$ in $L^1(\mathbb{R}^N)$ and $\|\nabla u_n\|_1 = V(u_n) \to V(u_0)$, and up to a subsequence, $u_n \to u_0$ a.e. on $\mathbb{R}^N$. Now letting $v_n := \frac{u_n}{\|u_n\|_1 + \|\nabla u_n\|_1} \in W^{1,1}(\mathbb{R}^N) \setminus \{0\}$ for large $n \in \mathbb{N}$, we see by the convergences of u_n and Fatou's lemma,

$$D_{\alpha,q} - \varepsilon < \|u_0\|_1 + \alpha \|u_0\|_q^q \leq \liminf_{n\to\infty} \left(\|v_n\|_1 + \alpha \|v_n\|_q^q \right) \leq \limsup_{n\to\infty} \left(\|v_n\|_1 + \alpha \|v_n\|_q^q \right) \leq \tilde{D}_{\alpha,q},$$

which implies $D_{\alpha,q} \leq \tilde{D}_{\alpha,q}$ since ε is arbitrary. Thus $D_{\alpha,q} = \tilde{D}_{\alpha,q}$ has been proved.

Next, we prove that $\tilde{D}_{\alpha,q}$ is not attained for all $\alpha > 0$. By Proposition 2.1 (iii), $\tilde{GN}_q$ is not attained, which yields

$$\tilde{GN}_q(u) < \tilde{GN}_q \quad \text{for all } u \in W^{1,1}(\mathbb{R}^N) \setminus \{0\}. \tag{3.1}$$

By contradiction, assume that $\tilde{D}_{\alpha,q}$ is attained by $u_0 \in W^{1,1}(\mathbb{R}^N) \setminus \{0\}$ with $\|u_0\|_1 + \|\nabla u_0\|_1 = 1$. Then using Proposition 2.1 (i), Proposition 2.2, $D_{\alpha,q} = \tilde{D}_{\alpha,q}$ and (3.1), we have

$$\begin{aligned}
D_{\alpha,q} = \tilde{D}_{\alpha,q} &= \|u_0\|_1 + \alpha\|u_0\|_q^q < \|u_0\|_1 + \alpha\,\tilde{GN}_q\|u_0\|_1^{q-(q-1)N}\|\nabla u_0\|_1^{(q-1)N} \\
&= \|u_0\|_1 + \alpha\, GN_q\|u_0\|_1^{q-(q-1)N}V(u_0)^{(q-1)N} \\
&= \frac{\|u_0\|_1\,(\|u_0\|_1 + V(u_0))^{q-1} + \alpha\, GN_q\|u_0\|_1^{q-(q-1)N}V(u_0)^{(q-1)N}}{(\|u_0\|_1 + V(u_0))^q} \\
&= \frac{\left(1+\frac{V(u_0)}{\|u_0\|_1}\right)^{q-1} + \alpha\, GN_q\left(\frac{V(u_0)}{\|u_0\|_1}\right)^{(q-1)N}}{\left(1+\frac{V(u_0)}{\|u_0\|_1}\right)^{q}} = f_\alpha\left(\frac{V(u_0)}{\|u_0\|_1}\right) \le \sup_{t>0} f_\alpha(t) = D_{\alpha,q},
\end{aligned}$$

which is a contradiction. Proof of Theorem 1.3 is complete. □

Lemma 3.2. *Let $1 < q \le N'$ and $\alpha > 0$. Assume that $D_{\alpha,q}$ is attained by $u_0 \in BV(\mathbb{R}^N)\setminus\{0\}$ with $\|u_0\|_1 + V(u_0) = 1$. Then there exist $R > 0$ and $x_0 \in \mathbb{R}^N$ such that u_0 is written as*

$$u_0 = \frac{N}{\omega_{N-1}R^{N-1}(N+R)}\chi_{B_R(x_0)}.$$

Proof. By Proposition 2.2 and the definition of GN_q, we see

$$\begin{aligned}
\sup_{t>0} f_\alpha(t) = D_{\alpha,q} &= \|u_0\|_1 + \alpha\|u_0\|_q^q \le \|u_0\|_1 + \alpha\, GN_q\|u_0\|_1^{q-(q-1)N}V(u_0)^{(q-1)N} \\
&= \frac{\|u_0\|_1\,(\|u_0\|_1 + V(u_0))^{q-1} + \alpha\, GN_q\|u_0\|_1^{q-(q-1)N}V(u_0)^{(q-1)N}}{(\|u_0\|_1 + V(u_0))^q} \\
&= \frac{\left(1+\frac{V(u_0)}{\|u_0\|_1}\right)^{q-1} + \alpha\, GN_q\left(\frac{V(u_0)}{\|u_0\|_1}\right)^{(q-1)N}}{\left(1+\frac{V(u_0)}{\|u_0\|_1}\right)^{q}} = f_\alpha\left(\frac{V(u_0)}{\|u_0\|_1}\right) \le \sup_{t>0} f_\alpha(t),
\end{aligned}$$

which implies that u_0 is a maximizer of GN_q. Then by Proposition 2.1 (ii), we can write $u_0 = \lambda\chi_{B_R(x_0)}$ for some $\lambda \in \mathbb{R}\setminus\{0\}$, $R > 0$ and $x_0 \in \mathbb{R}^N$. Moreover, since $\|u_0\|_1 = \lambda R^N\frac{\omega_{N-1}}{N}$ and $V(u_0) = \lambda R^{N-1}\omega_{N-1}$, the normalization $\|u_0\|_1 + V(u_0) = 1$ gives $\lambda = \frac{N}{\omega_{N-1}R^{N-1}(N+R)}$. Thus Lemma 3.2 has been proved. □

Proposition 3.3. *Let $1 < q \le N'$ and $\alpha > 0$. Assume that $\sup_{t>0} f_\alpha(t)$ admits a unique maximal point $t_0 > 0$. Then for each $x_0 \in \mathbb{R}^N$, the function*

$$\frac{t_0^N}{\omega_{N-1}N^{N-1}(1+t_0)}\chi_{B_{\frac{N}{t_0}}(x_0)} \tag{3.2}$$

is a maximizer of $D_{\alpha,q}$. Moreover, the function (3.2) is a unique maximizer of $D_{\alpha,q}$ except for the translation.

Proof. Let $v \in BV(\mathbb{R}^N)\setminus\{0\}$ be a maximizer of GN_q. Then Proposition 2.1 (ii) implies $v = \lambda\chi_{B_R(x_0)}$ for some $\lambda \in \mathbb{R}\setminus\{0\}$, $R > 0$ and $x_0 \in \mathbb{R}^N$. By the assumption, there exists a maximal point $t_0 > 0$ such that $\sup_{t>0} f_\alpha(t) = f_\alpha(t_0)$. Take $\lambda_0 > 0$ satisfying $\lambda_0^{\frac{1}{N}}\frac{V(v)}{\|v\|_1} = t_0$, i.e.,

$$\lambda_0 = \left(\frac{\|v\|_1}{V(v)}t_0\right)^N = \left(\frac{R}{N}t_0\right)^N, \tag{3.3}$$

where we used $\|v\|_1 = \lambda R^N \frac{\omega_{N-1}}{N}$ and $V(v) = \lambda R^{N-1}\omega_{N-1}$. Let $v_{\lambda_0}(x) := \lambda_0 v(\lambda_0^{\frac{1}{N}} x)$ and

$$w_{\lambda_0}(x) := \frac{v_{\lambda_0}(x)}{\|v_{\lambda_0}\|_1 + V(v_{\lambda_0})} = \frac{\lambda_0 v(\lambda_0^{\frac{1}{N}} x)}{\|v\|_1 + \lambda_0^{\frac{1}{N}} V(v)}.$$

Then by Proposition 2.2, we see

$$\sup_{t>0} f_\alpha(t) = D_{\alpha,q} \geq \|w_{\lambda_0}\|_1 + \alpha \|w_{\lambda_0}\|_q^q = \frac{\|v\|_1}{\|v\|_1 + \lambda_0^{\frac{1}{N}} V(v)} + \alpha \frac{\lambda_0^{q-1}\|v\|_q^q}{\left(\|v\|_1 + \lambda_0^{\frac{1}{N}} V(v)\right)^q}$$
$$= \frac{\left(1 + \lambda_0^{\frac{1}{N}} \frac{V(v)}{\|v\|_1}\right)^{q-1} + \alpha\, GN_q \left(\lambda_0^{\frac{1}{N}} \frac{V(v)}{\|v\|_1}\right)^{(q-1)N}}{\left(1 + \lambda_0^{\frac{1}{N}} \frac{V(v)}{\|v\|_1}\right)^q} = f_\alpha\left(\lambda_0^{\frac{1}{N}} \frac{V(v)}{\|v\|_1}\right) = f_\alpha(t_0) = \sup_{t>0} f_\alpha(t),$$

which implies that w_{λ_0} is a maximizer of $D_{\alpha,q}$. Moreover, by (3.3), we can compute

$$w_{\lambda_0} = \frac{t_0^N}{\omega_{N-1} N^{N-1}(1+t_0)} \chi_{B_{\frac{N}{t_0}}(\frac{N}{Rt_0}x_0)}.$$

Hence, the function (3.2) and its translations are maximizers of $D_{\alpha,q}$.

Next, we prove that the function (3.2) is a unique maximizer of $D_{\alpha,q}$ except for the translation. Assume that $u_0 \in BV(\mathbb{R}^N)\setminus\{0\}$ is a maximizer of $D_{\alpha,q}$ with $\|u_0\|_1 + V(u_0) = 1$. Then by Lemma 3.2, we can write

$$u_0 = \frac{N}{\omega_{N-1} R^{N-1}(N+R)} \chi_{B_R(x_0)}$$

for some $R > 0$ and $x_0 \in \mathbb{R}^N$, and then by putting $s_0 := \frac{N}{R}$, we have

$$u_0 = \frac{s_0^N}{\omega_{N-1} N^{N-1}(1+s_0)} \chi_{B_{\frac{N}{s_0}}(x_0)}.$$

To compete the proof of Proposition 3.3, it is enough to show $s_0 = t_0$. On the contrary, assume $s_0 \neq t_0$. Noting that u_0 is a maximizer both of $D_{\alpha,q}$ and GN_q, we see

$$\begin{aligned} D_{\alpha,q} &= \|u_0\|_1 + \alpha \|u_0\|_q^q = \|u_0\|_1 + \alpha\, GN_q \|u_0\|_1^{q-(q-1)N} V(u_0)^{(q-1)N} \\ &= \frac{\|u_0\|_1 (\|u_0\|_1 + V(u_0))^{q-1} + \alpha\, GN_q \|u_0\|_1^{q-(q-1)N} V(u_0)^{(q-1)N}}{(\|u_0\|_1 + V(u_0))^q} \\ &= \frac{\left(1 + \frac{V(u_0)}{\|u_0\|_1}\right)^{q-1} + \alpha\, GN_q \left(\frac{V(u_0)}{\|u_0\|_1}\right)^{(q-1)N}}{\left(1 + \frac{V(u_0)}{\|u_0\|_1}\right)^q} = f_\alpha\left(\frac{V(u_0)}{\|u_0\|_1}\right) = f_\alpha(s_0), \end{aligned}$$

where we used $s_0 = \frac{V(u_0)}{\|u_0\|_1}$, and thus it follows $D_{\alpha,q} = f_\alpha(s_0)$. Since t_0 is a unique maximal point of $\sup_{t>0} f_\alpha(t)$, we have by Proposition 2.2,

$$D_{\alpha,q} = \sup_{t>0} f_\alpha(t) = f_\alpha(t_0) > f_\alpha(s_0) = D_{\alpha,q},$$

which is a contradiction. Therefore, there holds $s_0 = t_0$. Thus Proposition 3.3 has been proved. □

Lemma 3.4. *Assume one of the following conditions*

(i) $1 < q < \frac{N+1}{N}$ *and* $\alpha > 0$, *(ii)* $q = \frac{N+1}{N}$ *and* $\alpha > \alpha_q^*$, *(iii)* $\frac{N+1}{N} < q < N'$ *and* $\alpha \geq \alpha_q^*$.

Then $\sup_{t>0} f_\alpha(t)$ *has a unique maximal point on* $(0,\infty)$.

Proof. Let $1 < q < N'$ and $\alpha > 0$. Then we can compute

$$\begin{aligned}&(1+t)^{q+1} f_\alpha'(t) = h(t)\\ &:= -(1+t)^{q-1} + \alpha\, GN_q (q-1) N t^{qN-(N+1)} - \alpha\, GN_q\,(N-q(N-1))\, t^{(q-1)N},\end{aligned}$$

and

$$\begin{aligned}&h'(t) = -(q-1)(1+t)^{q-2} - \alpha\, GN_q (q-1) N\,((N+1)-qN)\, t^{(q-1)N-2}\\ &- \alpha\, GN_q (q-1) N\,(N-q(N-1))\, t^{(q-1)N-1}.\end{aligned}$$

(i) Let $1 < q < \frac{N+1}{N}$ and $\alpha > 0$. In this case, since $qN-(N+1) < 0$, we obtain $\lim_{t\downarrow 0} h(t) = +\infty$, $\lim_{t\to+\infty} h(t) = -\infty$ and $h'(t) < 0$ for $t > 0$. Hence, $f_\alpha' = 0$ on $(0,\infty)$ has a unique solution $t_0 > 0$, and thus f_α is strictly increasing on $(0,t_0)$, and f_α is strictly decreasing on (t_0,∞). As a result, $\sup_{t>0} f_\alpha(t)$ has a unique maximal point t_0.

(ii) Let $q = \frac{N+1}{N}$ and $\alpha > \alpha_q^*$. In this case, we have

$$h(t) = -(1+t)^{\frac{1}{N}} + \alpha\, GN_q - \frac{\alpha\, GN_q}{N} t \quad \text{and} \quad h'(t) = -\frac{1}{N}(1+t)^{\frac{1}{N}-1} - \frac{\alpha\, GN_q}{N}.$$

Since $\alpha_q^* = \frac{1}{GN_q}$ by Proposition 2.2, we see $\lim_{t\downarrow 0} h(t) = -1 + \alpha\, GN_q > -1 + \alpha_q^* GN_q = 0$, $\lim_{t\to+\infty} h(t) = -\infty$ and $h'(t) < 0$ for $t > 0$. Hence, $f_\alpha' = 0$ on $(0,\infty)$ has a unique solution $t_0 > 0$, and thus f_α is strictly increasing on $(0,t_0)$, and f_α is strictly decreasing on (t_0,∞). As a result, $\sup_{t>0} f_\alpha(t)$ has a unique maximal point t_0.

(iii) Let $\frac{N+1}{N} < q < N'$. In this case, we observe $\lim_{t\downarrow 0} h(t) = -1$ and $\lim_{t\to+\infty} f_\alpha(t) = 0$. Computing

$$g(t) = \frac{(1+t)^{q-1}}{t^{qN-(N+1)}} \quad \text{and} \quad g'(t) = \frac{(1+t)^{q-2}}{t^{(q-1)N}}\left((N-q(N-1))\,t - (qN-(N+1))\right),$$

we see that $\inf_{t>0} g(t)$ has a unique minimal point $t_0 := \frac{qN-(N+1)}{N-q(N-1)} > 0$. We first consider the case $\alpha = \alpha_q^*$. By Proposition 2.2, we observe that $\inf_{t>0} g(t) = g(t_0)$ is equivalent to $f_{\alpha_q^*}(t_0) = 1$. As a result, we can conclude that $\sup_{t>0} f_{\alpha_q^*}(t) = 1$ has a unique maximal point t_0. Next, we consider the case $\alpha > \alpha_q^*$. In this case, we have $\alpha > \alpha_q^* = \frac{1}{GN_q} \inf_{t>0} g(t) = \frac{1}{GN_q} g(t_0)$, which implies $f_\alpha(t_0) > 1$. Hence, $\sup_{t>0} f_\alpha(t)$ has a maximal point on $(0,\infty)$. For proving the uniqueness of the maximal point of $\sup_{t>0} f_\alpha(t)$, we introduce $g_\beta(t)$ with $\beta > 1$ by

$$g_\beta(t) := \frac{(\beta t + \beta - 1)(1+t)^{q-1}}{t^{(q-1)N}}.$$

We observe that there holds $\lim_{t\downarrow 0} g_\beta(t) = \lim_{t\to+\infty} g_\beta(t) = +\infty$, and g_β is a strictly convex function on $(0,\infty)$. Therefore, for each $\beta > 1$, we see that $g_\beta = \alpha\, GN_q$ has at most two solutions on $(0,\infty)$. On the other hand, since $g_\beta = \alpha\, GN_q$ is equivalent to $f_\alpha = \beta$, we can conclude that the maximal point of $\sup_{t>0} f_\alpha(t)$ is unique. Thus Lemma 3.4 has been proved. $\square$

Proof of Theorem 1.4. Proposition 3.3 together with Lemma 3.4 implies the assertion of Theorem 1.4. $\square$

References

[1] M. Ishiwata, *Existence and nonexistence of maximizers for variational problems associated with Trudinger-Moser type inequalities in* $\mathbb{R}^N$, Math. Ann. **351** (2011), 781–804.

[2] M. Ishiwata and H. Wadade, *On the effect of equivalent constraints on a maximizing problem associated with the Sobolev type embeddings in* $\mathbb{R}^N$. Math. Ann. **364** (2016), no. 3-4, 1043–1068.